1 Maxwell's Equations I

I will start with a brief overview of the fundamental equations that underlie all electromagnetic phenomena: Maxwell's equations. I will try not to drift too much into historical details, even if they are fascinating and often throw light on interpretational issues. Rather, I will assume that you already have studied the basics of electromagnetism from an introductory physics course and that you know about Coulomb's law, some of the historical underpinnings, and so on. I will not waste time either on reviewing vector calculus in detail, as one of the prerequisites for this course is Mathematical Physics I, in which you are supposed to have studied the topic.

Most likely you have seen before Maxwell's equations in integral form,

\[
\oint_S E \cdot da = \frac{Q_{\text{in}}}{\epsilon_0},
\]

(1)

\[
\oint_S B \cdot da = 0,
\]

(2)

\[
\oint_C E \cdot dr = -\frac{d}{dt} \int_S B \cdot da,
\]

(3)

\[
\oint_C B \cdot dr = \mu_0 I + \mu_0 \epsilon_0 \frac{d}{dt} \int_S E \cdot da.
\]

(4)

Eq. (1) is Gauss's law, which says that the flux of the electric field through any closed surface \( S \) is proportional to the net charge \( Q_{\text{in}} \) inside. (See fig. 1.) So, it tells us that charges are sources of electric fields.

Eq. (2) is Gauss's law for magnetic fields: the magnetic flux through an arbitrary closed surface is zero. Every magnetic field line that enters the region bounded by the surface \( S \) must leave eventually. In other words, the field lines don't start or end at any
point inside the region and since $S$ is arbitrary, magnetic field lines don't start or end at any point in space, period. Thus, as far as anyone knows, \textit{there are no magnetic charges (also know as magnetic monopoles) in nature}. (Fig. 2.)

![Figure 1](image)

\textbf{Figure 1}

Eq. (3) is the integral form of Faraday's law: the line integral of the electric field along a closed curve $C$ (that is, the induced emf if the curve is a wire) equals the negative of the time rate of change of the magnetic flux ($\Phi_B = \oint_S \mathbf{B} \cdot d\mathbf{a}$) through a surface $S$ whose boundary is $C$. (See fig. 3. below.) This means that \textit{changing magnetic fluxes are also sources of electric fields}.

![Figure 2](image)

\textbf{Figure 2}

Finally, equation 4 is Ampère-Maxwell's law: the piece $\oint_C \mathbf{B} \cdot d\ell = \mu_0 I$ is Ampère's law, which basically tells us that \textit{electric currents are sources of magnetic fields}. The second term on the right-hand side, $\mu_0 I_d$, where $I_d = \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{a}$ is Maxwell's displacement current accounts for the fact that \textit{changing electric fluxes are also sources of magnetic fields}. 
To these, we need to add the Lorentz's force law, which specifies how the charges are affected by the fields,

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \]  

That's it! All of classical electromagnetism is contained in these equations. They contain the dynamics of the fields, how charge distributions and electric currents produce electric and magnetic fields and, in turn, how these fields affect the motion of charges. The rest of this course (and the next) will be an elucidation of the consequences of these equations under a variety of conditions, some very interesting, some very practical, some fascinating, and some... well, not so much.

The first step into new–and possibly unknown to you–territory is to recast Maxwell's equations in differential form. This makes them more manageable for many applications (in fact, they are simpler) and provides us with extra physical insight. But before we do it, let me remind you of some important definitions and theorems from vector calculus.
1.2 A Vector Calculus Interlude

You must have seen (and surely remember!) the definitions of the vector differential operations gradient, divergence and curl. Most likely what stuck in your memory is their Cartesian expressions using the \( \nabla \) ("del" or "nabla") operator

\[
\nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.
\]

Let \( \mathbf{r} = x \hat{x} + y \hat{y} + z \hat{z} \). If \( V = V(\mathbf{r}) \) is a scalar field and \( \mathbf{f} = f(\mathbf{r}) = f_x(\mathbf{r}) \hat{x} + f_y(\mathbf{r}) \hat{y} + f_z(\mathbf{r}) \hat{z} \), a vector field, then, in Cartesian coordinates,

\[
\text{grad } f = \nabla V = \frac{\partial V}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z},
\]

\[
\text{div } f = \nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z},
\]

\[
\text{curl } f = \nabla \times f = \text{det} \left( \begin{array}{ccc}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
f_x & f_y & f_z
\end{array} \right)
\]

\[
\left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) \hat{x} + \left( \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) \hat{y} + \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \hat{z}.
\]

These expressions are valid only in Cartesian coordinates. You know, however, that it is perfectly all right (and often convenient and even necessary) to use other coordinate systems, such as spherical polar coordinates, cylindrical coordinates, confocal ellipsoidal coordinates (I am showing off), etc. What you may not remember (or may have never seen) is that there are coordinate-independent definitions of these operations, which provide crucial insight into their meaning. In fact, take \( \mathbf{r} \) to be the position vector in some arbitrary coordinate system. Given a scalar field \( V \), its differential is

\[
dV \equiv \text{grad } V \cdot d\mathbf{r}.
\]

This defines the gradient of \( V \), and it is valid regardless of the coordinate system employed.

For the divergence, we define

\[
\text{div } f \equiv \lim_{V \to 0} \frac{1}{V} \int_{S} f \cdot d\mathbf{a}.
\]
And for the curl,

\[ \hat{n} \cdot \text{curl} f = \lim_{A \to 0} \frac{1}{A} \oint_C f \cdot dr. \]  

(Note: It is common—although not universal—practice to use the notation \( \nabla V \) for \( \text{grad} \ V \), \( \nabla \cdot f \) for \( \text{div} f \), and \( \nabla \times f \) for \( \text{curl} f \), even if the coordinate system is not Cartesian. We will follow this convention, but you should be careful not to confuse them with their Cartesian expressions when working in other coordinate systems.)

The interpretation of definitions 6, 7, and 8 is as follows. The change of a scalar field \( V \) from \( r \) to \( r + dr \) is \( dV = \nabla V \cdot dr \). Take the infinitesimal displacement \( dr \) to be along the surface \( V = \text{const} \). Then \( dV = 0 \), and therefore \( \nabla V \) is perpendicular to the surface of constant \( V \). On the other hand, for an arbitrary infinitesimal displacement \( dr \), the angle \( \theta \) between \( \nabla V \) and \( dr \) is given by \( dV = |\nabla V| |dr| \cos \theta \), and so \( dV \) is a "maximum" for \( \cos \theta = 1 \), (that is, \( \theta = 0 \)) which implies that \( dr \) is parallel to \( \nabla V \). Therefore, \( \nabla V \) points in the direction that maximizes the change of \( V \). And, of course, \( |\nabla V| \) is the rate of change of \( V \) in that direction.

As you know from your calculus courses, the differential of a scalar function \( V \) is

\[ dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 
\left( \frac{\partial \hat{V}}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z}), \]

and we recover the usual Cartesian expression \( \text{grad} \ V = \nabla V = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} \).

Now to equation 7. The flux \( \oint_S f \cdot ds \) through the closed surface \( S \) quantifies "how much" of the field \( f \) enters or leaves the region bounded by \( S \). Take a point \( P \) within that region. As we make the region shrink toward \( P \) the area of \( S \) decreases and (in general) so does the flux. So, generically, we would expect the integral to tend to zero as the region shrinks to a point, since all the points on \( S \) are so close to each other when the region is sufficiently small that \( f \) changes very little from point to point. Dividing by the volume \( V \) of the region we are effectively computing a sort of limiting flux per unit volume, which may or may not be finite when \( V \) tends to zero. If it is finite or zero then there is a well-defined divergence of the field at that point.

One way of visualizing the meaning of divergence is by looking at fluid flow. Choose the vector field \( f \) to be the mass flux \( \rho v \) of the fluid, where \( \rho = \rho(r, t) \) is the mass density and \( v = v(r, t) \), the velocity of the fluid at a given point \( P \). (The mass flux has dimensions of mass per unit area per unit time, as you can easily check.) Take now
a closed surface \( S \) (containing \( P \)) that the fluid is crossing. The scalar product of the mass flux and an infinitesimal area element \( da \) on \( S \) is the (infinitesimal) flux \( d\Phi \) of the fluid through \( da \),

\[
d\Phi = (\rho v) \cdot da,
\]

which measures the mass of fluid per unit time crossing \( da \). (Check the dimensions!) So,

\[
\Phi = \int_S (\rho v) \cdot da
\]

is the mass of fluid per unit time coming out of the region bounded by \( S \). Our definition of divergence tells us that

\[
\nabla \cdot (\rho v) = \lim_{V \to 0} \frac{1}{V} \int_S (\rho v) \cdot da,
\]

which means that \( \nabla \cdot (\rho v) \) is the density (mass per unit volume) of fluid emerging from \( P \) per unit time. If \( \nabla \cdot (\rho v) > 0 \), then the fluid density is flowing outward from the infinitesimal region around \( P \). In this case \( P \) is a "source" of fluid. If \( \nabla \cdot (\rho v) < 0 \), the fluid density is flowing into the region, in which case \( P \) is a "sink".

One more comment. The mass per unit time per unit volume of fluid going into or out of the region is \((dm/dt)/\text{vol} = d\rho/dt\). If \( d\rho/dt > 0 \), then the density is increasing over time (fluid flowing into the region); \( d\rho/dt < 0 \) for a net density escaping (outward flow) the region; and zero for no net change. If we shrink the region until it becomes an infinitesimal neighborhood of \( P \), then mass conservation (no sources or sinks present) and our interpretation above of the divergence of \( \rho v \) imply

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho v) = 0
\]

This is known as a **continuity equation**, in this case for fluid flow. Continuity equations are very important in physics as they constitute the mathematical expression of conserved quantities.

Since electric charge is also conserved, there should be a corresponding continuity equation for it. Define the (average) magnitude of the charge current density \( J \) as the charge flowing per unit area (normal to the current) per unit time,

\[
(J) \equiv \frac{\Delta q}{\text{area} \times \Delta t}
\]
For charges moving in the $x$ direction, an area patch perpendicular to the current can be taken to be $\Delta y \Delta z$. So,

$$J = \frac{\Delta q}{(\Delta y \Delta z) \times \Delta t} = \frac{\Delta q \Delta x}{\Delta x (\Delta y \Delta z) \times \Delta t} = \frac{\Delta q \Delta x}{\text{vol} \times \Delta t} = \frac{\Delta q}{\text{vol}} v = \rho v,$$

where $v$ is the average speed of the charges and $\rho$ is the charge density. (Careful: this $\rho$ is not the same as the one for the fluid!) In the limit of infinitesimal volume and time, $J$ becomes the instantaneous current density, and we can write it in vector form as

$$J = \rho v.$$  \hfill (10)

The continuity equation for charge conservation should then be

$$\frac{d\rho}{dt} + \nabla \cdot J = 0$$  \hfill (11)

As a matter of fact this equation, although fundamental, doesn't need to be postulated. It is a direct consequence of Maxwell's equations! OK, don't be impatient, I will prove it to you in a little while.

Now, what about the Cartesian expression for divergence? Can we derive it from our general definition? Yes, we can. Choose an infinitesimal rectangular box with faces parallel to the coordinate planes and centered on $P$, as shown in the figure below.

We want to compute the flux per unit volume of the field $f$ through the box. Why? Because it is in the definition of divergence.

![Figure 5](https://example.com/Figure5.png)

The flux per unit volume through the faces parallel to the $XZ$ plane is
Similarly, the flux per unit volume for the faces parallel to the $YZ$ plane is $\sum f_x \hat{e}_x \sum x$ and, for the faces parallel to the $XY$ plane, it is $\sum f_y \hat{e}_y \sum y$. Adding them up we get the total flux per unit volume,

$$\frac{\Phi_{XZ}}{\text{vol}} = \frac{\left[f_y(y + dy) - f_y(y)\right] dx \ dz}{dx \ dy \ dz} = \frac{\left[f_y(y + dy) - f_y(y)\right]}{dy} = \frac{\partial f_y}{\partial y}.$$ 

But, according to our definition (see Eq. 7) this total flux per unit volume for the infinitesimal box is the divergence of $f$. So, we found the expression for $\nabla \cdot f$ in Cartesian coordinates. When the time comes, I will derive the expression for the divergence in other coordinate systems.

Finally, let us analyze the meaning of curl. Let me repeat (almost verbatim) what I said earlier when discussing the divergence, but suitably modified. The line integral $\oint_C f \cdot d\mathbf{r}$ through the closed curve $C$ (also known as the circulation of $f$) quantifies the amount of "rotation"—or circulation—of the field around the two-dimensional region bounded by $C$. Take a point $P$ within that region. As we make the region shrink toward $P$ the length of $C$ decreases and (in general) so does the circulation. So, generically, we would expect the integral to tend to zero as the region shrinks to a point, since all the points on $C$ are so close to each other that $f$ changes very little from point to point. Dividing by the area $A$ we are effectively computing a sort of limiting circulation per unit area, which may or may not be finite when $A$ tends to zero. If it is finite (or zero) then there is a well-defined curl of the field at that point. One more thing: our definition is for the component of the curl in the direction normal to the surface, $\hat{n} \cdot \nabla \times f$. This is as it should be, since the computed circulation per unit area is a scalar, not a vector.

As before, let's take look at fluid flow. The curl of the velocity field $\mathbf{v}(\mathbf{r})$ of the fluid at a given point is a measure of how much the fluid "circulates" around the point. This is known as vorticity. If there is a vortex, a whirlpool, of fluid around $P$, then $\nabla \times \mathbf{v} \neq 0$. In such a case, an object floating on the region where the curl is non-zero will tend to start rotating.

To derive the Cartesian form of the curl, choose a rectangular region on the $YZ$ plane, centered on the point $P(x = 0, y, z)$. In Cartesian coordinates,

$$\mathbf{f} \cdot d\mathbf{r} = f_x \ dx + f_y \ dy + f_z \ dz.$$ 

The component of the curl in the direction of the positive $x$ axis is
Therefore, if we integrate in the direction indicated in the figure (counterclockwise as seem from the positive x axis), the normal \( \hat{n} \) to the surface coincides with \( \hat{x} \) and we are in business.

For the chosen rectangle, \( dx = 0 \), so \( \mathbf{f} \cdot d\mathbf{r} = f_y \, dy + f_z \, dz \). There are four contributions to the circulation, at points 1 - 4. These are

\[
\oint \mathbf{f} \cdot d\mathbf{r} = f_y \left(0, y, z - \frac{1}{2} \, dz\right) \, dy - f_y \left(0, y, z + \frac{1}{2} \, dz\right) \, dy - f_z \left(0, y - \frac{1}{2} \, dy, z\right) \, dz + f_z \left(0, y - \frac{1}{2} \, dy, z\right) \, dz
\]

The negative sign in front of the second \( f_y \) comes from point 4, because the integration there is in the negative y direction. Similar comments apply to the negative sign in front of the first \( f_z \). The partial derivative of \( f_y \) respect to \( z \) is

\[
\frac{\partial f_y}{\partial z} = \lim_{\Delta z \to 0} \frac{f_y(x, y, z + \Delta z) - f_y(x, y, z)}{\Delta z}
\]

which justifies writing

\[
f_y(x, y, z + dz) = f_y(x, y, z) + \frac{\partial f_y}{\partial z} \, dz
\]

This lets us compute all the terms above in the line integral,
\[
\int f \cdot dr = \left[ f_x(0, y, z) - \frac{1}{2} \frac{\partial f_y}{\partial z} dz \right] dy - \left[ f_y(0, y, z) + \frac{1}{2} \frac{\partial f_y}{\partial z} dz \right] dy + \left[ f_z(0, y, z) - \frac{1}{2} \frac{\partial f_z}{\partial y} dy \right] dz - \left[ f_z(0, y, z) + \frac{1}{2} \frac{\partial f_z}{\partial y} dy \right] dz
\]

which, after cancellations, simplifies to

\[
\int f \cdot dr = \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) dy dz
\]

From the definition of curl, and for our rectangle, \( \mathbf{\hat{x}} \cdot (\nabla \times f) = \lim_{A \to 0} \frac{1}{A} \oint_C f \cdot dr = \frac{1}{dy dz} \oint_C f \cdot dr \), which implies that

\[
(\nabla \times f)_x = \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}.
\]

A completely analogous derivation with rectangles sitting on the \( XZ \) plane and the \( XY \) plane gives, respectively,

\[
(\nabla \times f)_y = \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \quad \text{and} \quad (\nabla \times f)_z = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}.
\]

That's it. (Wasn't that fun?!)

Moving on, let me remind you of a couple of very important theorems of vector calculus. First is the divergence (or Gauss's) theorem:

\[
\int_V \nabla \cdot f \, d\tau = \oint_S f \cdot d\mathbf{a} \tag{12}
\]

![Figure 7](10_01_Maxwell's Equations I.nb)
As you can see, a volume integral over some region $\mathcal{V}$ is related to an integral over the surface boundary $\mathcal{S}$ of the region.

Then there is **Stokes' theorem**:

$$
\int_{\mathcal{S}} (\nabla \times \mathbf{f}) \cdot d\mathbf{a} = \oint_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{r}
$$

(13)

![Figure 8](image)

In this case an integral over a surface $\mathcal{S}$ is equated to an integral over the boundary $\mathcal{C}$ of the surface.

Given our definitions of curl and div, the proof of the theorems are almost trivial. In an infinitesimal region of volume $d\tau$ the (infinitesimal) flux of $\mathbf{f}$ through the boundary of the region is (see Eq. 7),

$$
\int_{\delta\mathcal{S}} \mathbf{f} \cdot d\mathbf{a} = \nabla \cdot \mathbf{f} \ d\tau
$$

(Note: The symbol $\delta\mathcal{S}$ is to remind you that this is an infinitesimal area, and that the integral is also infinitesimal.) This looks almost like the divergence theorem, except that we need to convert it from an infinitesimal to a finite volume. We do it by noticing that any finite region can be decomposed into an infinite number of infinitesimal blocks, as shown in figure 9. The (finite) volume integral is just the sum of the contributions of every block, $\int_{\mathcal{V}} \nabla \cdot \mathbf{f} \ d\tau$. For the surface integral, there is a subtlety. Look at the adjacent blocks 1 and 2. The flux through the right face of block 1 is equal in magnitude but opposite in sign to the flux through the left face of block 2. (For block 1 the field is coming out of the block, while for 2 it is coming into the block.) So, when adding the flux contributions, adjacent faces cancel out. This is true for every pair of adjacent faces on adjacent blocks, so the net contribution to the flux comes only from those faces that are not paired up and their fluxes canceled, such as the right face of
block 2 in the figure. These faces correspond to the outer surface of the finite region, and so the total flux is just what's expected, $\oint_S f \cdot da$. This proves the theorem. ■

Figure 9

To prove Stokes' theorem, we can do something similar. We take the surface $S$ bounded by the curve $C$, and subdivide it into infinitesimal rectangular cells (figure 10).

Figure 10

From the definition of curl, the (infinitesimal) circulation around one infinitesimal cell is

$$\oint_{Sc} f \cdot dr = (\hat{n} \cdot \nabla \times f) da = (\nabla \times f) \cdot da$$

When adding the circulations (closed line integrals) from all the cells, the contributions from adjacent sides of adjacent cells cancel each other out. This is so because the direction of integration for the adjacent sides is opposite, as is evident from the figure. What remains then is the contributions from unpaired sides, which correspond to the curve $C$, and the sum equals the circulation $\oint_C f \cdot dr$. For the curl piece, the addition is
even simpler: we just add all the scalar products to get the surface integral \( \int_S (\nabla \times f) \cdot d\mathbf{a} \).}

## 1.3 Maxwell's Equations in Differential Form

With the divergence and Stokes' theorems in hand, we can now rewrite Maxwell's equations. Start from Gauss's law and apply the divergence theorem:

\[
\frac{Q_{\text{in}}}{\varepsilon_0} = \oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} \, d\tau.
\]

But \( Q_{\text{in}} = \int_V \rho \, d\tau \), where \( \rho = \rho(\mathbf{r}) \) is the charge density. So,

\[
\int_V \left( \frac{1}{\varepsilon_0} \rho - \nabla \cdot \mathbf{E} \right) \, d\tau = 0
\]

which must be valid for any region \( \mathcal{V} \). The only way for this to be true is if the integrand is itself zero, which means that

\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho. \tag{14}
\]

An analogous derivation starting from \( \oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \) tells us that

\[
\nabla \cdot \mathbf{B} = 0. \tag{15}
\]

Let's work now on Faraday's law. Assuming that the surface area remains constant, we can bring the time derivative inside the integral,

\[
- \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} = \int_S - \frac{d\mathbf{B}}{dt} \cdot d\mathbf{a}.
\]

But Stoke's theorem tells us that

\[
\oint_C \mathbf{E} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{a}.
\]

Thus, from Faraday's law (Eq. 3),

\[
\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = \int_S - \frac{d\mathbf{B}}{dt} \cdot d\mathbf{a},
\]

which has to be valid for any surface \( S \). Therefore,
Finally, Ampère-Maxwell's law. The current \( I \) is related to the current density \( \mathbf{J} \) through \( I = \int \mathbf{J} \cdot d\mathbf{a} \). Following steps similar to those used with Faraday's law we find (do it!)

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\]

(Eq. 17)

Equations 14-17 are Maxwell's equations in differential form. The interpretation of these equations is facilitated by the meaning of div and curl we discussed above. Eq. (14) tells us that electric fields have a scalar source: electric charges (charge densities or charge distributions). Eq. (15) similarly tells us that magnetic fields have no scalar source, they never start and/or end at any point. So, no magnetic monopoles in nature. Eq. (16) implies that electric fields can also have a vector source, a changing magnetic field, in which case their curl is non-zero, a property that gives the field a different character than in electrostatics. (It is non-conservative. More about this later.) Finally, Eq. (17) provides vector sources for magnetic fields. This come in two flavors: electric currents (current density \( \mathbf{J} \)) and/or changing electric fields (also known as the displacement current term).

I promised before that I would prove the conservation of electric charge (continuity equation) from Maxwell's equations. It goes like this. Start by taking the time derivative of Gauss's law in differential form,

\[
\frac{d}{dt} \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \frac{d\rho}{dt}
\]

(Eq. 18)

Reversing the order of the operators \( d/dt \) and \( \nabla \) and using eq. (17),

\[
\frac{1}{\varepsilon_0} \frac{d\rho}{dt} = \nabla \cdot \left( \frac{d\mathbf{E}}{dt} \right) = \frac{1}{\mu_0 \varepsilon_0} \nabla \cdot (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \Rightarrow \frac{d\rho}{dt} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{B}) - \nabla \cdot \mathbf{J}.
\]

(Eq. 19)

But \( \nabla \cdot (\nabla \times \mathbf{f}) = 0 \), for any vector field \( \mathbf{f} \). So, \( d\rho/dt = -\nabla \cdot \mathbf{J} \).
1.4 A Note on Units

Unlike in classical mechanics, the equations of electromagnetic theory look slightly different depending on the system of units employed. There are three systems in use and none can really be said to be superior in all respects to the others. It depends on the applications. It is then desirable for you to have at least a passing acquaintance with all three systems. We will use the *Système International d'Unités* (International System of Units, international abbreviation SI, http://www.bipm.fr/en/si/; see also physics.nist.gov/cuu/Units/), because it is the one used by most introductory textbooks, and recommended by the International Union of Pure and Applied Physics (http://www.iupap.org/index.html). The base units of this system are the meter (length), kilogram (mass), second (time), ampere (electric current), kelvin (temperature), mole (amount of substance), and candela (luminous intensity). In the SI system the (derived) unit of charge is the coulomb.

The other two systems, Gaussian and Heaviside, are also widely used. Please, see Appendix C of Griffiths' book (or check any other textbook) for an explanation and conversion factors between SI and Gaussian and Heaviside.

1.5 Electric-Magnetic Duality

Let's rewrite Maxwell's equations in vacuum ($\rho = 0$, $\mathbf{J} = 0$) and in Gaussian units to make them look more symmetrical:

\[
\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\n\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}.
\]

Written like this we see that there is a remarkable symmetry between electric and magnetic fields. In fact, if we make the duality transformation, $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$, the equations remain invariant. (The upper set becomes the lower set, and vice versa.) This is known as electric-magnetic duality which, as we can see, is a property of Maxwell's equations in vacuum. However, this doesn't work if charges and currents are present. (Try it!) In order to restore the duality symmetry we would need to include magnetic charge densities $\rho_m$ associated to magnetic charges $q_m$ (monopoles), and
magnetic currents $J_m$. The generalized Maxwell's equations would then look like this (in Gaussian units, again):

\[
\nabla \cdot E = 4 \pi \rho_e, \quad \nabla \times E = -(4 \pi / c) J_m - (1 / c) \partial B / \partial t \label{eq:21}
\]
\[
\nabla \cdot B = 4 \pi \rho_m, \quad \nabla \times B = +(4 \pi / c) J_e + (1 / c) \partial E / \partial t.
\]

where $\rho_e$ and $J_e$ are the electric charge density and current, respectively, and $\rho_m$ and $J_m$ are the magnetic charge density and current. The full duality transformations that leave these equations invariant are

\[
E \rightarrow B, \quad B \rightarrow -E, \\
\rho_e \rightarrow \rho_m, \quad \rho_m \rightarrow -\rho_e, \\
J_e \rightarrow J_m, \quad J_m \rightarrow -J_e. \label{eq:22}
\]

You should check that, in fact, equations (21) remain invariant under the transformation (22). A direct consequence of eqs. (21) is the conservation of electric and magnetic charge, expressed in their corresponding continuity equations

\[
d \rho / dt = -\nabla \cdot J \label{eq:23}
\]

where $J = J_e \lor J_m$. Why did nature choose Maxwell's equations without magnetic charges rather than their extended counterpart (21) is still a mystery. Some (for now hypothetical) recent extensions of our well-established theories of the fundamental interactions in nature predict the existence of magnetic monopoles, but no experimental evidence has ever been found for them. Even the notion of duality has become important in recent speculations regarding the basic laws of nature. Time will tell.

\section{1.6 Maxwell's Equations for Static Fields}

Maxwell's equations when the fields don't change over time are

\[
\nabla \cdot E = \frac{1}{\varepsilon_0} \rho, \label{eq:24}
\]
\[
\nabla \cdot B = 0, \label{eq:25}
\]
\[
\nabla \times E = 0, \label{eq:26}
\]
\[
\nabla \times B = \mu_0 J. \label{eq:27}
\]

Equations (20) and (22) are Maxwell's equations for electrostatics. If we make the assumption that the electric field at an infinite distance from all sources is zero, they are equivalent to Coulomb's law. (In fact, we will prove this statement later.) Similarly,
eqs. (21) and (23) are Maxwell's equations for magnetostatics, and they are equivalent to the Biot-Savart law under the assumption that the magnetic field vanishes at an infinite distance from any currents. A major portion of this course will be devoted to a study of the solutions to these equations, since they have important physical applications.

The fact that, in electrostatics, $\nabla \times \mathbf{E} = 0$ implies that the electric field can be written as the gradient of a scalar function

$$\mathbf{E} = -\nabla V. \quad (28)$$

[This is so on account of the identity $\nabla \times (\nabla \phi) = 0$, valid for any scalar field $\phi$.] The scalar field $V$ is known as the electric potential (or just potential). Dimensionally, electric potential is potential energy per unit charge.

The fact that the curl of the electrostatic field $\mathbf{E}$ is zero is equivalent to the statement that the line integral of $\mathbf{E}$ over any closed path is zero: $\oint \mathbf{E} \cdot d\mathbf{r} = 0$. You may recall that the work of any conservative over a closed path is always zero. Well, this is precisely what I am saying here. The work done per unit charge done by an electrostatic field is zero for any closed path. In other words, the electrostatic field is conservative. The very existence of a potential function $V$ tells us the same. Whenever a vector field is conservative, there is an associated scalar potential, and vice versa.

As I explain in more detail in section 1.7, there is an ambiguity in the definition of the potential. In the case of electrostatic fields, this ambiguity means that we can add any constant to the potential without changing the value of the electric field. (This is obvious, since $\mathbf{E}$ is the gradient of $V$.) In more elementary treatments we say that what we measure are only differences in potential energy, and that we need always to fix a "zero level" of potential energy. (The standard convention is to make $V = 0$ at infinity.)

The advantage of using $V$ instead of $\mathbf{E}$ comes from the fact that, in general, it is easier to manipulate scalars than vectors. Moreover, in terms of the potential $V$, Maxwell's equation (26) has nothing to say. (It is just an identity.) Equation (24), on the other hand, becomes $\nabla \cdot (\nabla V) = \frac{-1}{\varepsilon_0} \rho$, or

$$\nabla^2 V = -\frac{1}{\varepsilon_0} \rho. \quad (29)$$

This is Poisson's equation. In the case of no charge densities present, $\rho = 0$, and

$$\nabla^2 V = 0, \quad (30)$$

which is Laplace's equation. Both equations are extremely important in electrostatics (physics, in general) and we will study methods for solving them in this course. In fact,
solving either of them to find the potential is equivalent to solving both Maxwell's equations for electrostatics, (24) and (26). [Having found $V$ we simply use (28) to determine $E$.] So, you see, we have traded two vector first-order partial differential equations [(24) and (26)] for one second-order partial differential equation, either (29) or (30). We will have more to say about the electric potential later when start the discussion of electrostatics in earnest.

In the case of the magnetic field, the curl is not necessarily zero, but the divergence always is. Using the identity $\nabla \cdot (\nabla \times f) = 0$, valid for any vector function $f$, we can write the magnetic field as

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (31)$$

The vector field $\mathbf{A}$ is called the magnetic vector potential (or simply the vector potential). Plugging (31) into (27),

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J},$$

and using the identity $\nabla \times (\nabla \times f) = \nabla (\nabla \cdot f) - \nabla^2 f$, valid for any vector function $f$,

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Once again there is an ambiguity in the definition of $\mathbf{A}$. We can add the gradient of any scalar field $\psi$ to $\mathbf{A}$ without affecting $\mathbf{B}$. (Because $\nabla \times \nabla \psi = 0$.) Because of that we are free to choose, for example, an $\mathbf{A}$ whose divergence is zero, $\nabla \cdot \mathbf{A} = 0$, turning the above equation into

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \quad (32)$$

Eq. (32) is three Poisson's equations, one for each component of $\mathbf{A}$. Although not as useful as the Poisson equation for the electric potential, eq. (32) together with (31) could be used as an alternative foundation for magnetostatics.

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1.7 Potentials, Gauge Transformations, and Electrodynamics

What if the fields are not static? Can we still write them in terms of potentials? Well, obviously nothing changes for the magnetic field ($\nabla \cdot \mathbf{B} = 0$, no matter what). But the electric field is no longer "curless". Nevertheless, since $\mathbf{B} = \nabla \times \mathbf{A}$, writing

$$\mathbf{E} \equiv -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad (33)$$
we get
\[ \nabla \times \mathbf{E} = -\nabla \times \nabla V - \nabla \times \left( \frac{\partial \mathbf{A}}{\partial t} \right) = - \frac{\partial}{\partial t} \nabla \times \left( \frac{\partial \mathbf{A}}{\partial t} \right) = - \frac{\partial \mathbf{B}}{\partial t}. \tag{34} \]

So, we see that \( \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \) (Faraday's law) and \( \nabla \cdot \mathbf{B} = 0 \) (no monopoles) are automatically satisfied with the definitions (31) and (33). [Eq. (33) reduces to (28) for a static \( \mathbf{A} \).] Checking Gauss's law,
\[ \nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \]
which is satisfied if
\[ -\nabla^2 V - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \frac{1}{\varepsilon_0} \rho. \tag{36} \]

The Ampère-Maxwell law, \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t \), becomes
\[ \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right). \tag{37} \]

Using again the identity \( \nabla \times (\nabla \times \mathbf{f}) = \nabla (\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f} \),
\[ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{38} \]

We can make everything more symmetrical and easier to handle by adding a term \( \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} \) to both sides of (36) and rearranging,
\[ \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{1}{\varepsilon_0} \rho + \frac{\partial}{\partial t} \left( \nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right), \tag{39} \]
and rewriting (38) as
\[ \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \nabla \left( \nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right). \tag{40} \]

Eqs. (39) and (40) both contain the piece \( \nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \partial V/\partial t \) which, as we will see, can be made equal to zero without altering Maxwell's equations. In fact, definitions (31) and (33) give us the freedom to make the changes to the potentials
\[ \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \psi, \tag{41} \]
and
\[ V \rightarrow V' = V - \frac{\partial \psi}{\partial t}, \tag{42} \]
without altering \( \mathbf{E} \) and \( \mathbf{B} \). Checking,
\[ B = \nabla \times A \rightarrow B' = \nabla \times (A + \nabla \psi) = \nabla \times A + \nabla \times \nabla \psi = B, \]  
(43)

and

\[ E = -\nabla V - \frac{\partial A}{\partial t} \rightarrow E' = -\nabla \left( V - \frac{\partial \psi}{\partial t} \right) - \frac{\partial}{\partial t} \left( A + \nabla \psi \right) = -\nabla V + \nabla \frac{\partial \psi}{\partial t} - \frac{\partial A}{\partial t} - \frac{\partial}{\partial t} \nabla \psi. \]  
(44)

But \( \nabla (\partial \psi / \partial t) = (\partial / \partial t) \nabla \psi \). So, \( E' = E \).

The transformations (41) and (42) are known as \textbf{gauge transformations}. These are the general statements for the "ambiguities" in the definition of \( V \) and \( A \) that we mentioned in the previous section. The invariance of electromagnetic theory under gauge transformations is an example of a \textbf{gauge symmetry}. This "gauge freedom" allows us to choose \textit{any} scalar function \( \psi \) and transform the potentials according to (41) and (42) without affecting any physical quantities (the one we measure with our instruments). Thus, in classical electromagnetic theory (what we are studying in this course), the potentials \( A \) and \( V \) are assumed to have no absolute physical significance, since we can always redefine them by a gauge transformation without altering the underlying physics. This is not the case in quantum mechanics, though. As you will see when you study the topic, potentials do have a more physical (as opposed to just mathematically convenient) role to play. (For example, in the Bohm-Aharonov effect.)

With the gauge freedom afforded by (42) and (43), we can choose a specific scalar function \( \psi \) that will make \( \nabla \cdot A + \mu_0 \varepsilon_0 \partial V / \partial t \) equal to zero. (Choosing a particular \( \psi \) is known as \textbf{fixing the gauge} or choosing \textbf{a gauge condition}.) The gauge condition

\[ \nabla \cdot A + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 \]  
(45)

is known as the \textbf{Lorentz gauge}. In general, \( A \) and \( V \) do not satisfy (45), but we can always make a gauge transformation that will ensure that the new fields \( A' \) and \( V' \) do satisfy it. Using the Lorentz gauge, eqs. (39) and (40) become

\[ \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{1}{\varepsilon_0} \rho, \]  
(46)

and

\[ \mu_0 \varepsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J, \]  
(47)

respectively. Equation (46) is the \textbf{inhomogeneous wave equation} for \( V \), and (47) the inhomogeneous wave equation for \( A \). They are direct consequences of the Gauss and
Ampère-Maxwell laws and are very important.

Equations (46) and (47), together with (31) and (33) which define the electric and magnetic fields in terms of $A$ and $V$, are equivalent to the full Maxwell equations and form the foundation for the study of electrodynamics (next semester!). This is analogous to the way that eqs. (29) and (32), together with (28) and (31), are equivalent to Maxwell's equations for static fields, and constitute the foundation of electrostatics and magnetostatics (this semester!).